C^*-CROSSED PRODUCTS BY PARTIAL ACTIONS AND ACTIONS OF INVERSE SEMIGROUPS

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Abstract

The recently developed theory of partial actions of discrete groups on C^*-algebras is extended. A related concept of actions of inverse semigroups on C^*-algebras is defined, including covariant representations and crossed products. The main result is that every partial crossed product is a crossed product by a semigroup action.

1. Introduction

Recently the notion of a partial crossed product of a C^*-algebra by a discrete group was defined by McClanahan [4] as a generalization of Exel's definition in [2]. The more well-established notion of the crossed product of a C^*-algebra by an action of a group uses a homomorphism into the automorphism group of the C^*-algebra. The idea of a partial action is to replace the automorphism group by the inverse semigroup of partial automorphisms. A partial automorphism is an isomorphism between two closed ideals of a C^*-algebra. Of course we cannot talk about a homomorphism from a group into an inverse semigroup; a partial action is an appropriate generalization. In Section 2 we give a detailed discussion of partial actions.

After replacing the automorphism group by the semigroup of partial automorphisms the next natural step is to replace our group by an inverse semigroup. This makes it possible to use a more natural semigroup homomorphism instead of a partial action. We develop the elementary theory of an action of an inverse semigroup in Section 4, and we define the crossed product by an inverse semigroup action in Section 5.

It turns out that there is a close connection between partial crossed products and crossed products by inverse semigroup actions. In Section 6 we explore this connection, showing that every partial crossed product is isomorphic to a crossed product by an inverse semigroup action.

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2. Partial actions

In this section we discuss the notion of *partial actions* defined by McClanahan [4] which is a generalization of Exel's definition in [2]. The major new result is Theorem 2.6.

Definition 2.1. Let A be a C^*-algebra. A *partial automorphism* of A is a triple (α, I, J) where I and J are closed ideals in A and $\alpha : I \to J$ is a $*$-isomorphism. We are going to write α instead of (α, I, J) if the domain and range of α are not important.

If (α, I, J) and (β, K, L) are partial automorphisms of A then the product $\alpha\beta$ is defined as the composition of α and β with the largest possible domain, that is, $\alpha\beta : \beta^{-1}(I) \to A$, $\alpha\beta(a) = \alpha(\beta(a))$. It is clear that $\beta^{-1}(I)$ is a closed ideal of K. Since a closed ideal of a closed ideal of A is also a closed ideal of A, the product $(\alpha\beta, \beta^{-1}(I), \alpha\beta(\beta^{-1}(I)))$ is a partial automorphism too.

A semigroup S is an *inverse semigroup* if for every $s \in S$ there exists a unique element s^* of S so that $ss^*s = s$ and $s^*ss^* = s^*$. The map $s \mapsto s^*$ is an involution. An element $f \in S$ satisfying $f^2 = f$ is called an *idempotent* of S, and in this case $f = f^*$ also. The set of idempotents of an inverse semigroup is a semilattice with $e \wedge f = ef$. Our general reference on semigroups is [3]. It is easy to see that the set $\mathrm{PAut}(A)$ of partial automorphisms of A is a unital inverse semigroup with identity (ι, A, A), where ι is the identity map on A, and $(\alpha, I, J)^* = (\alpha^{-1}, J, I)$.

Definition 2.2. Let A be a C^*-algebra and G be a discrete group with identity e. A *partial action* of G on A is a collection $\{(\alpha_s, D_{s^{-1}}, D_s) : s \in G\}$ of partial automorphisms (denoted by α or by (A, G, α)) such that

(i) $D_e = A$

(ii) α_{st} extends $\alpha_s\alpha_t$, that is, $\alpha_{st}|\alpha_t^{-1}(D_{s^{-1}}) = \alpha_s\alpha_t$ for all $s, t \in G$.

Proposition 2.3. *If α is a partial action of G on A then*

(i) *α_e is the identity map ι on A*

(ii) *$\alpha_{s^{-1}} = \alpha_s^{-1}$ for all $s \in G$.*

Proof. The statements follow from the following two identities

$$\iota = \alpha_e\alpha_e^{-1} = \alpha_{ee}\alpha_e^{-1} = \alpha_e\alpha_e\alpha_e^{-1} = \alpha_e \ ,$$

$$\alpha_s\alpha_{s^{-1}} = \alpha_{ss^{-1}}|D_s = \alpha_e|D_s = \iota|D_s \ .$$

□

Lemma 2.4. *If α is a partial action of G on A then $\alpha_t(D_{t^{-1}}D_s) = D_t D_{ts}$ for all $s, t \in G$.*

Proof. By Proposition 2.3 (ii), $\alpha_t(D_{t^{-1}}D_s) = \alpha_{t^{-1}}^{-1}(D_{t^{-1}}D_s) = \alpha_{t^{-1}}^{-1}(D_s)$ which is the domain of $\alpha_{s^{-1}}\alpha_{t^{-1}}$ and hence is contained in the domain $D_{(s^{-1}t^{-1})^{-1}} = D_{ts}$ of $\alpha_{s^{-1}t^{-1}}$. Since the range of α_t is D_t, we have

$$\alpha_t(D_{t^{-1}}D_s) \subset D_t D_{ts} \qquad \text{for all} \quad s, t \in G.$$

Since α_t is an isomorphism, the containment above implies

$$D_{t^{-1}}D_s \subset \alpha_t^{-1}(D_t D_{ts}) = \alpha_{t^{-1}}(D_t D_{ts}) \qquad \text{for all} \quad s, t \in G.$$

Replacing t by t^{-1} and s by ts gives

$$D_t D_{ts} \subset \alpha_t(D_{t^{-1}}D_{t^{-1}ts}) = \alpha_t(D_{t^{-1}}D_s) \qquad \text{for all} \quad s, t \in G$$

as desired. □

Lemma 2.5. *If α is a partial action of G on A then $\alpha_t(D_{t^{-1}}D_{s_1} \cdots D_{s_n}) = D_t D_{ts_1} \cdots D_{ts_n}$ for all $t, s_1, \ldots, s_n \in G$.*

Proof. The statement follows from the following calculation using Lemma 2.4:

$$\begin{aligned} \alpha_t(D_{t^{-1}}D_{s_1} \cdots D_{s_n}) &= \alpha_t(D_{t^{-1}}D_{s_1} \cdots D_{t^{-1}}D_{s_n}) \\ &= \alpha_t(D_{t^{-1}}D_{s_1}) \cap \cdots \cap \alpha_t(D_{t^{-1}}D_{s_n}) \\ &= D_t D_{ts_1} \cap \cdots \cap D_t D_{ts_n} \\ &= D_t D_{ts_1} \cdots D_{ts_n}. \end{aligned}$$

□

Theorem 2.6. *If α is a partial action of G on A then the partial automorphism $\alpha_{s_1} \cdots \alpha_{s_n}$ has domain $D_{s_n^{-1}} D_{s_n^{-1}s_{n-1}^{-1}} \cdots D_{s_n^{-1}\cdots s_1^{-1}}$ and range $D_{s_1}D_{s_1s_2} \cdots D_{s_1\cdots s_n}$ for all $s_1, \ldots, s_n \in G$.*

Proof. The theorem is proven inductively. The statement is clear for $n = 1$. The induction step follows from the following calculation using Lemma 2.5:

$$\begin{aligned} \text{domain } \alpha_{s_1} \cdots \alpha_{s_n} &= \alpha_{s_n}^{-1}(\text{ domain } \alpha_{s_1} \cdots \alpha_{s_{n-1}}) \\ &= \alpha_{s_n^{-1}}(D_{s_n} D_{s_{n-1}^{-1}} \cdots D_{s_{n-1}^{-1}\cdots s_1^{-1}}) \\ &= D_{s_n^{-1}} D_{s_n^{-1}s_{n-1}^{-1}} \cdots D_{s_n^{-1}\cdots s_1^{-1}}. \end{aligned}$$

The other part now follows since the range of $\alpha_{s_1} \cdots \alpha_{s_n}$ is the domain of $\alpha_{s_n^{-1}} \cdots \alpha_{s_1^{-1}}$. □

Corollary 2.7. *The conditions in the definition of a partial action can be reformulated as follows:*
(i) $D_e = A$
(ii)′ $\alpha_{st}|D_{t^{-1}}D_{t^{-1}s^{-1}} = \alpha_s\alpha_t$.

Remark 2.8. If α is a partial action of G on A then α generates a unital inverse subsemigroup $S = \{\alpha_{s_1} \cdots \alpha_{s_n} : s_1, \ldots, s_n \in G, n \in \mathbf{Z}\}$ of the semigroup of partial automorphisms of A. Theorem 2.6 tells us the domains and ranges.

3. Covariant representation

We continue our discussion of partial actions. Here the major new results are Theorem 3.8 and Corollary 3.10.

Definition 3.1. Let α be a partial action of G on A. A *covariant representation* of α is a triple (π, u, H) where $\pi : A \to B(H)$ is a nondegenerate representation of A on the Hilbert space H and $g \mapsto u_g : G \to B(H)$, where u_g is a partial isometry on H with initial space $\pi(D_{g^{-1}})H$ and final space $\pi(D_g)H$, such that
(i) $u_g\pi(a)u_{g^{-1}} = \pi(\alpha_g(a))$ for all $a \in D_{g^{-1}}$
(ii) $u_{st}h = u_su_th$ for all $h \in \pi(D_{t^{-1}}D_{t^{-1}s^{-1}})H$.

Notice that by the Cohen-Hewitt factorization theorem [1], $\pi(D_g)H$ is a closed subspace of H and so the notations for the initial and final spaces make sense.

Proposition 3.2. *If (π, u, H) is a covariant representation then $u_e = 1_H$ (the identity map on H) and $u_{s^{-1}} = u_s^*$ for all $s \in G$.*

Proof. Since π is a nondegenerate representation, H is the closed span of $\pi(A)H = \pi(D_e)H$. But since $\pi(D_e)H$ is a closed subspace of H we have $H = \pi(D_e)H$. Hence u_e has initial and final space H which means u_e is a unitary on H. By Definition 3.1 (ii) we have $u_e = u_eu_e$ which implies $1_H = u_eu_e^{-1} = u_eu_eu_e^{-1} = u_e$.

For the second statement we have to show that $\langle u_sh, k\rangle = \langle h, u_{s^{-1}}k\rangle$ for all $h, k \in H$. We can write $h = h_1 + h_2$ where $h_1 \in \pi(D_{s^{-1}})H$, $h_2 \in (\pi(D_{s^{-1}})H)^\perp$ and $k = k_1 + k_2$ where $k_1 \in \pi(D_s)H$, $k_2 \in (\pi(D_s)H)^\perp$. So it suffices to show that

$$\langle u_sh_1, k_1 + k_2\rangle = \langle h_1 + h_2, u_{s^{-1}}k_1\rangle.$$

Since $\langle u_sh_1, k_2\rangle = 0 = \langle h_2, u_{s-1}k_1\rangle$, it suffices to show that

$$\langle u_sh_1, k_1\rangle = \langle h_1, u_{s^{-1}}k_1\rangle \qquad \text{for all} \quad h_1 \in \pi(D_{s-1})H, \quad k_1 \in \pi(D_s)H.$$

Since $h_1 = u_{s^{-1}}l$ for some $l \in \pi(D_s)H$ it remains to show that

$$\langle u_su_{s^{-1}}l, k_1\rangle = \langle u_{s^{-1}}l, u_{s^{-1}}k_1\rangle \qquad \text{for all} \quad k_1, l \in \pi(D_s)H.$$

By Definition 3.1 (ii) we have $u_em = u_su_{s^{-1}}m$ for all $m \in \pi(D_s)H$. Hence $u_su_{s^{-1}}l = l$ and the statement follows from the fact that $u_{s^{-1}}$ is a partial isometry with initial space $\pi(D_s)H$. ∎

Let $\pi_u : A \to B(H_u)$ be the universal representation of a C^*-algebra A on the universal Hilbert space H_u. If I is an ideal of A then the double dual I^{**} of I, identified with the strong operator closure of $\pi_u(I)$, is an ideal of the enveloping von Neumann algebra A^{**} of A, which is identified with the strong operator closure of $\pi_u(A)$. As such, I^{**} has the form pA^{**} for some central projection p in A^{**}.

Definition 3.3. Let α be a partial action of G on A. For $s \in G$, p_s denotes the central projection of A^{**} which is the identity of D_s^{**}.

Let (π, u, H) be a covariant representation of (A, G, α). Since π is a nondegenerate representation of A, π can be extended to a normal morphism of A^{**} onto $\pi(A)''$. We will denote this extension also by π.

Lemma 3.4. *Let π be a representation of A on H, I be a closed ideal of A and p be the central projection of A^{**} which is the identity of I^{**}. Then $\pi(I)H = \pi(p)H$.*

Proof. If $a \in I$ and $h \in H$ then $\pi(a)h = \pi(pa)h = \pi(p)\pi(a)h \in \pi(p)H$ which implies $\pi(I)H \subset \pi(p)H$.

On the other hand $\pi(p)$ is in the strong operator closure of $\pi(I)$ and hence $\pi(p)$ is the strong operator limit of a net $\{\pi(a_\lambda)\}$ in $\pi(I)$. Hence $\|\pi(p)h - \pi(a_\lambda)h\| \to 0$ for all $h \in H$. Since $\pi(I)H$ is closed, this means $\pi(p)h \in \pi(I)H$ for all $h \in H$. □

Corollary 3.5. *Let π be a representation of A on H. Then*

$$\pi(D_{s_1} \cdots D_{s_n})H = \pi(p_{s_1} \cdots p_{s_n})H$$

for all $s_1, \ldots, s_n \in G$.

Proof. It is clear that $p_{s_1} \cdots p_{s_n}$ is the identity of $(D_{s_1} \cdots D_{s_n})^{**}$ in A^{**}, so the statement follows from Lemma 3.4. □

Corollary 3.6. *If (π, u, H) is a covariant representation then $u_s u_s^* = \pi(p_s)$ and $u_s^* u_s = \pi(p_{s^{-1}})$ for all $s \in G$.* □

Proposition 3.7. *If (π, u, H) be a covariant representation then for $g_1, \ldots, g_n \in G$ we have*

$$\begin{aligned} u_{g_1} \cdots u_{g_n} u_{g_n}^* \cdots u_{g_1}^* &= \pi(p_{g_1} \cdots p_{g_1 \cdots g_n}) \\ u_{g_n}^* \cdots u_{g_1}^* u_{g_1} \cdots u_{g_n} &= \pi(p_{g_n^{-1}} \cdots p_{g_n^{-1} \cdots g_1^{-1}}) \end{aligned}$$

Proof. The second equality follows from the first one taking conjugates. For $n = 1$ the first equality is true by Corollary 3.6. Applying induction we get

$$\begin{aligned} u_{g_1} \cdots u_{g_n} u_{g_n}^* \cdots u_{g_1}^* &= u_{g_1} u_{g_1}^* u_{g_1} \circ \pi(p_{g_2} \cdots p_{g_2 \cdots g_n}) \circ u_{g_1}^* \\ &= u_{g_1} \circ \pi(p_{g_1^{-1}} p_{g_2} \cdots p_{g_2 \cdots g_n}) \circ u_{g_1}^* \\ &= \pi(\alpha_{g_1}(p_{g_1^{-1}} p_{g_2} \cdots p_{g_2 \cdots g_n})) \\ &= \pi(p_{g_1} \cdots p_{g_1 \cdots g_n}), \end{aligned}$$

by Lemma 2.5. Notice that we extended α_{g_1} to get an isomorphism $\alpha_{g_1} : D^{**}_{s^{-1}} \to D^{**}_s$ between the double duals. □

Theorem 3.8. *Let (π, u, H) be a covariant representation. Then $u_{s_1} \cdots u_{s_n}$ is a partial isometry with initial and final spaces*

$$\pi(D_{s_n^{-1}} D_{s_n^{-1} s_{n-1}^{-1}} \cdots D_{s_n^{-1} \cdots s_1^{-1}})H \quad \text{and} \quad \pi(D_{s_1} D_{s_1 s_2} \cdots D_{s_1 \cdots s_n})H,$$

for all $s_1, \ldots, s_n \in G$.

Proof. It is clear from Proposition 3.7 that $u_{s_1} \cdots u_{s_n}$ is a partial isometry. The statement about the initial and final spaces follows from Corollary 3.5. □

Corollary 3.9. *If (π, u, H) is a covariant representation then*

$$u_{s_1 \cdots s_n} h = u_{s_1} \cdots u_{s_n} h \quad \text{for all} \quad h \in \pi(D_{s_n^{-1}} D_{s_n^{-1} s_{n-1}^{-1}} \cdots D_{s_n^{-1} \cdots s_1^{-1}})H$$

and

$$\pi(a) u_{s_1 \cdots s_n} = \pi(a) u_{s_1} \cdots u_{s_n} \quad \text{for all} \quad a \in D_{s_1} D_{s_1 s_2} \cdots D_{s_1 \cdots s_n}.$$

Proof. The first statement follows by induction using Definition 3.1 (ii) and the fact that

$$D_{s_n^{-1}} D_{s_n^{-1} s_{n-1}^{-1} \cdots s_1^{-1}} \supset D_{s_n^{-1}} D_{s_n^{-1} s_{n-1}^{-1}} \cdots D_{s_n^{-1} \cdots s_1^{-1}}.$$

By the first statement we have

$$u_{s_n^{-1} \cdots s_1^{-1}} \pi(a^*) = u_{s_n^{-1}} \cdots u_{s_1^{-1}} \pi(a^*),$$

and the second statement follows from this by taking conjugates. □

Corollary 3.10. *If (π, u, H) is a covariant representation, then*

$$S = \{u_{s_1} \cdots u_{s_n} : s_1, \ldots, s_n \in G\}$$

is a unital inverse semigroup of partial isometries of H.

The situation is more delicate than it may appear at first glance: the following example shows that a set of partial isometries with commuting initial and final projections does not necessarily generate an inverse semigroup of partial isometries.

Example 3.11. Let $\alpha \in (0, \frac{\pi}{2})$ and

$$U = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

be partial isometries on $\mathbf{C}^3$. A short calculation shows that U^2, V^2, UV, VU are all partial isometries and so all the initial and final projections of U and V commute. But

$$W = (UV)^2 = \begin{pmatrix} 0 & -\sin\alpha\cos^3\alpha & \sin^2\alpha\cos^2\alpha \\ 0 & \cos^4\alpha & -\sin\alpha\cos^3\alpha \\ 0 & 0 & 0 \end{pmatrix}$$

is not a partial isometry because

$$WW^*W = \begin{pmatrix} 0 & -\sin\alpha\cos^7\alpha & \sin^2\alpha\cos^6\alpha \\ 0 & \cos^8\alpha & -\sin\alpha\cos^7\alpha \\ 0 & 0 & 0 \end{pmatrix} \neq W.$$

This example is a modification of an idea of Marcelo Laca.

4. Action of an inverse semigroup

In this section we define an action of a unital inverse semigroup and a covariant representation of such an action. The assumption of the identity of the semigroup is for technical reasons. In the absence of an identity we can easily add one. There is a connection between crossed products by partial actions and crossed products by semigroup actions, which we are going to explore in Section 6.

Definition 4.1. Let A be a C^*-algebra and S be a unital inverse semigroup with identity e. An *action* of S on A is a semigroup homomorphism $s \mapsto (\beta_s, E_{s^*}, E_s) : S \to \mathrm{PAut}(A)$, with $E_e = A$.

Notice that $\beta_{s^*} = \beta_s^{-1}$ for all $s \in S$ so the notations E_{s^*} and E_s make sense. It can be shown as in Proposition 2.3 that β_e is the identity map ι on A. Also if $f \in S$ is an idempotent then so is β_f, which means β_f is the identity map on $E_{f^*} = E_f$.

Lemma 4.2. *If β is an action of the unital inverse semigroup S on A then $\beta_t(E_{t^*}E_s) = E_{ts}$ for all $s, t \in S$.*

Proof. The proof follows from the following calculation:

$$\beta_t(E_{t^*}E_s) = \beta_{t^*}^{-1}(E_{t^*}E_s) = \beta_{t^*}^{-1}(E_s) = \text{domain } \beta_{s^*}\beta_{t^*} = \text{domain } \beta_{s^*t^*} = E_{ts}.$$

□

Example 4.3. We have seen in Remark 2.8 that a partial action α of a group G on A generates a unital inverse subsemigroup $S = \{\alpha_{s_1} \cdots \alpha_{s_n} : s_1, \ldots, s_n \in G, n \in \mathbf{Z}\}$ of the semigroup $\mathrm{PAut}(A)$ of partial automorphisms on A, and hence determines an action $s \mapsto \beta_s = s$ of the inverse semigroup S on A.

There is a much more important inverse semigroup action associated with α, which we will define in Section 6 and is based upon the construction used in the following proposition.

Proposition 4.4. *Let α be a partial action of a group G on the C^*-algebra A, and let (π, u, H) be a covariant representation of α . Let $S = \{(\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n}) : g_1, \ldots, g_n \in G\}$. Then S is a unital inverse semigroup with coordinatewise multiplication. For $s = (\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n}) \in S$ let*

$$\begin{aligned} E_{s^*} &= D_{g_n^{-1}} D_{g_n^{-1} g_{n-1}^{-1}} \cdots D_{g_n^{-1} \cdots g_1^{-1}} \\ E_s &= D_{g_1} D_{g_1 g_2} \cdots D_{g_1 \cdots g_n} \\ \beta_s &= \alpha_{g_1} \cdots \alpha_{g_n} : E_{s^*} \to E_s. \end{aligned}$$

Then β is an action of S on A.

Proof. By Remark 2.8 and Corollary 3.10, S is a unital inverse semigroup with identity (α_e, u_e), where e is the identity of G. It is clear that β is a semigroup homomorphism with $E_{(\alpha_e, u_e)} = D_e = A$. □

Definition 4.5. Let β be an action of the unital inverse semigroup S on A. A *covariant representation* of β is a triple (π, v, H) where $\pi : A \to B(H)$ is a nondegenerate representation of A on the Hilbert space H and $s \mapsto v_s$ is a semigroup homomorphism from S into an inverse semigroup of partial isometries on H such that

(i) $v_s \pi(a) v_{s^*} = \pi(\beta_s(a))$ for all $a \in E_{s^*}$
(ii) v_s has initial space $\pi(E_{s^*})H$ and final space $\pi(E_s)H$.

It can be shown similarly as in the group case (Proposition 3.2) that $v_e = 1_H$ and $v_{s^*} = v_s^*$. We denote the class of all covariant representations of (A, S, β) by $\mathrm{CovRep}(A, S, \beta)$.

Proposition 4.6. *Keeping the notations of Proposition 4.4 define $v : S \to B(H)$ by $v_s = u_{g_1} \cdots u_{g_n}$, where $s = (\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n})$. Then $(\pi, v, H) \in \mathrm{CovRep}(A, S, \beta)$. Furthermore if $(\rho, z, K) \in \mathrm{CovRep}(A, S, \beta)$ then the function*

$$w : G \to B(K) \quad \text{defined by} \quad w_g = z(\alpha_g, u_g)$$

gives a covariant representation (ρ, w, K) of (A, G, α). The connections can be visualized by the following diagram

$$\begin{array}{ccccc} & & G & & \\ & \swarrow u & \downarrow & w \searrow & \\ B(H) & \xleftarrow{v} & S & \xrightarrow{z} & B(K) \\ & \nwarrow \pi & & \rho \nearrow & \\ & & A & & \end{array}$$

Proof. It is clear that v is a semigroup homomorphism from S into an inverse semigroup of partial isometries on H. To check Definition 4.5 (i) let $s = (\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n}) \in S$ and $a \in E_{s^*} = D_{g_n^{-1}} D_{g_n^{-1} g_{n-1}^{-1}} \cdots D_{g_n^{-1} \cdots g_1^{-1}}$. Using Definition 3.1 (i) and Lemma 2.5 we have

$$\begin{aligned} v_s \pi(a) v_{s^*} &= \mathrm{Ad}\ u_{g_1} \cdots u_{g_n} \circ \pi(a) \\ &= \mathrm{Ad}\ u_{g_1} \cdots u_{g_{n-1}} \circ \pi \circ \alpha_{g_n}(a) \\ &= \cdots \\ &= \pi \circ \alpha_{g_1} \cdots \alpha_{g_n}(a) \\ &= \pi(\beta_s(a)). \end{aligned}$$

By Theorem 3.8 v_s has the desired initial and final spaces. For the second part of the theorem let $a \in D_{g^{-1}} = E_{s^*}$, where $s = (\alpha_g, u_g)$. Then

$$w_g \rho(a) w_{g^{-1}} = z_s \rho(a) z_{s^*} = \rho(\beta_s(a)) = \rho(\alpha_g(a)),$$

and so w satisfies Definition 3.1 (i). To check Definition 3.1 (ii) let $g_1, g_2 \in G$, $h \in \rho(D_{g_2^{-1}} D_{g_2^{-1} g_1^{-1}})K$ and let $s = (\alpha_{g_1 g_2}, u_{g_1 g_2}), s_1 = (\alpha_{g_1}, u_{g_1}), s_2 = (\alpha_{g_2}, u_{g_2}) \in S$. By Definition 2.2 (ii) $\alpha_{g_1 g_2}(\alpha_{g_1}\alpha_{g_2})^* = \alpha_{g_1}\alpha_{g_2}(\alpha_{g_1}\alpha_{g_2})^*$. By Definition 3.1 (ii) and Theorem 3.8 $u_{g_1 g_2}(u_{g_1} u_{g_2})^* = u_{g_1} u_{g_2}(u_{g_1} u_{g_2})^*$. Hence $s(s_1 s_2)^* = s_1 s_2 (s_1 s_2)^*$ and so $z_s z_{(s_1 s_2)^*} = z_{s_1 s_2} z_{(s_1 s_2)^*}$. Since the final space of $z_{(s_1 s_2)^*}$ is $\rho(D_{g_2^{-1}} D_{g_2^{-1} g_1^{-1}})K$, it follows that $z_s h = z_{s_1 s_2} h$. Thus

$$w_{g_1 g_2} h = z_s h = z_{s_1 s_2} h = z_{s_1} z_{s_2} h = w_{g_1} w_{g_2} h$$

as desired. It is clear that w_g has the required initial and final spaces. □

Notice that if in the previous theorem we let $z = v$ then the construction gives $w = u$.

Not every unital inverse semigroup action arises from a partial action via the construction of Proposition 4.4, as we can see in the next example.

Example 4.7. Let $S = \{e, f\}$ be the unital inverse semigroup that contains the identity e and an idempotent $f \neq e$. Let $A = \mathbf{C}$ and β_s be the identity map ι of A for all $s \in S$. Suppose there is a partial action (A, G, α) and a covariant representation (π, u, H) of α so that S can be identified with the inverse semigroup $\{(\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n}) : g_1, \ldots, g_n \in G\}$ and $\beta_s = \alpha_{g_1} \cdots \alpha_{g_n}$ for all $s = (\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n}) \in S$. Clearly e is identified with $(\iota, 1_H)$, where 1_H is the identity of $B(H)$. Suppose f is identified with $(\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n})$. By the definition of β_s, for all $g_1, \ldots, g_n \in G$ we have $\alpha_{g_1} \cdots \alpha_{g_n} = \iota$. Since f is an idempotent $u_{g_1} \cdots u_{g_n}$ is an idempotent too. Hence for all $h \in H$ we have

$$\begin{aligned} h &= \pi(1)(h) = \pi(\beta_f(1))(h) \\ &= (u_{g_1} \cdots u_{g_n} \pi(1) u_{g_1} \cdots u_{g_n})(h) \\ &= u_{g_1} \cdots u_{g_n}(h). \end{aligned}$$

This means that $u_{g_1} \cdots u_{g_n}$ must be the identity of $B(H)$. But this is a contradiction since e and f are different elements of S.

5. The crossed product

McClanahan[4] defines the *partial crossed product* $A \times_\alpha G$ of the C^*-algebra A and the group G by the partial action α as the enveloping C^*-algebra of $L = \{x \in l^1(G, A) : x(g) \in D_g\}$ with multiplication and involution

$$\begin{aligned} (x * y)(g) &= \sum_{h \in G} \alpha_h[\alpha_{h^{-1}}(x(h)) y(h^{-1}g)], \\ x^*(g) &= \alpha_g(x(g^{-1})^*). \end{aligned}$$

He shows that there is bijective correspondence $(\pi, u, H) \leftrightarrow (\pi \times u, H)$ between covariant representations of (A, G, α) and nondegenerate representations of $A \times_\alpha G$, where $\pi \times u$ is the extension of the representation of L defined by

$$x \mapsto \sum_{g \in G} \pi(x(g)) u_g.$$

We are going to follow his footsteps constructing the *crossed product* of a C^*-algebra and a unital inverse semigroup by an action β .

Let β be an action of the unital inverse semigroup S on the C^*-algebra A. Let

$$L = \{x \in l^1(S, A) : x(s) \in E_s\}$$

have the norm, scalar multiplication and addition inherited from $l^1(S, A)$. Define multiplication and involution on L by

$$\begin{aligned} (x * y)(s) &= \sum_{rt = s} \beta_r[\beta_{r^*}(x(r)) y(t)] \\ x^*(s) &= \beta_s(x(s^*)^*). \end{aligned}$$

Notice that by Lemma 4.2 $(x*y)(s) \in E_s$. A short calculation shows that $\|x*y\| \leq \|x\|\|y\|$ and so $x*y \in L$:

$$\begin{aligned}
\|x*y\| &= \sum_{s\in S}\sum_{rt=s} \|\beta_r(\beta_{r^*}(x(r))y(t))\| \\
&= \sum_{s\in S}\sum_{rt=s} \|\beta_{r^*}(x(r))y(t)\| \\
&\leq \sum_{r\in S}\sum_{t\in S} \|\beta_{r^*}(x(r))\|\|y(t)\| \\
&\leq \sum_{r\in S} \|\beta_{r^*}(x(r))\| \sum_{t\in S} \|y(t)\| \\
&\leq \|x\|\|y\|.
\end{aligned}$$

One can easily check that $\|x^*\| = \|x\|$ and so $x^* \in L$. We are going to denote by $a\delta_s$ the function in L taking the value a at s and zero at every other element of S. Notice that $a_s\delta_s * a_t\delta_t = \beta_s(\beta_{s^*}(a_s)a_t)\delta_{st}$ and $(a\delta_s)^* = \beta_{s^*}(a^*)\delta_{s^*}$.

Proposition 5.1. *L is a Banach *-algebra.*

Proof. Let $x, y, z \in L$ and $a \in \mathbf{C}$. Then it is easy to see that $(x+y)^* = x^* + y^*$ and $(ax)^* = \bar{a}x^*$. We also have $x^{**} = x$ since

$$x^{**}(s) = \beta_s(x^*(s^*)^*) = \beta_s(\beta_{s^*}(x(s^{**})^{**})) = x(s).$$

Next we show that $(x*y)^* = y^* * x^*$. It suffices to show this for $x = a_s\delta_s$ and $y = a_t\delta_t$. We have

$$\begin{aligned}
(a_s\delta_s * a_t\delta_t)^* &= (\beta_s(\beta_{s^*}(a_s)a_t)\delta_{st})^* \\
&= \beta_{t^*s^*}((\beta_s(\beta_{s^*}(a_s)a_t))^*)\delta_{t^*s^*} \\
&= \beta_{t^*s^*}(\beta_s(a_t^*\beta_{s^*}(a_s^*)))\delta_{t^*s^*} \\
&= \beta_{t^*}(a_t^*\beta_{s^*}(a_s^*))\delta_{t^*s^*} \\
&= \beta_{t^*}(\beta_t(\beta_{t^*}(a_t^*))\beta_{s^*}(a_s^*))\delta_{t^*s^*} \\
&= \beta_{t^*}(a_t^*)\delta_{t^*} * \beta_{s^*}(a_s^*)\delta_{s^*} \\
&= (a_t\delta_t)^* * (a_s\delta_s)^*
\end{aligned}$$

as desired.

Finally we show that $(x*y)*z = x*(y*z)$. Again it suffices to show this for $x = a_r\delta_r$, $y = a_s\delta_s$ and $z = a_t\delta_t$. If $\{u_\lambda\}$ is an approximate identity for E_{s^*}, then

we have

$$\begin{aligned}
(a_r\delta_r * a_s\delta_s) * a_t\delta_t &= \beta_r(\beta_{r^*}(a_r)a_s)\delta_{rs} * a_t\delta_t \\
&= \beta_{rs}(\beta_{s^*r^*}(\beta_r(\beta_{r^*}(a_r)a_s))a_t)\delta_{rst} \\
&= \lim_\lambda \beta_{rs}(\beta_{s^*}(\beta_{r^*}(a_r)a_s)u_\lambda a_t)\delta_{rst} \\
&= \lim_\lambda \beta_r(\beta_{r^*}(a_r)a_s\beta_s(u_\lambda a_t))\delta_{rst} \\
&= \lim_\lambda \beta_r(\beta_{r^*}(a_r)\beta_s(\beta_{s^*}(a_s)u_\lambda a_t))\delta_{rst} \\
&= \beta_r(\beta_{r^*}(a_r)\beta_s(\beta_{s^*}(a_s)a_t))\delta_{rst} \\
&= a_r\delta_r * \beta_s(\beta_{s^*}(a_s)a_t)\delta_{st} \\
&= a_r\delta_r * (a_s\delta_s * a_t\delta_t)
\end{aligned}$$

as desired. □

Definition 5.2. If $(\pi, v, H) \in \mathrm{CovRep}(A, S, \beta)$ then define $\pi \times v : L \to B(H)$ by

$$(\pi \times v)(x) = \sum_{s\in S} \pi(x(s))v_s.$$

Proposition 5.3. *$(\pi \times v)$ is a *-homomorphism onto the C^*-algebra*

$$C^*(\pi, v) = \overline{\sum_{s\in S} \pi(E_s)v_s}.$$

Proof. First notice that by Definition 4.5, $v_sv_{s^*}\pi(a_s) = \pi(a_s) = \pi(a_s)v_sv_{s^*}$ for all $a_s \in E_s$. If $s, t \in S$, $a \in E_s$ and $b \in E_t$ then

$$\begin{aligned}
\pi(a)v_s\pi(b)v_t &= v_s\pi(\beta_{s^*}(a))v_{s^*}v_s\pi(b)v_t \\
&= v_s\pi(\beta_{s^*}(a)b)v_t \\
&= v_s\pi(\beta_{s^*}\beta_s(\beta_{s^*}(a)b))v_t \\
&= v_sv_{s^*}\pi(\beta_s(\beta_{s^*}(a)b))v_sv_t \\
&= \pi(\beta_s(\beta_{s^*}(a)b))v_{st}
\end{aligned}$$

and

$$(\pi(a)v_s)^* = v_{s^*}\pi(a^*) = v_{s^*}v_s\pi(\beta_{s^*}(a^*))v_{s^*} = \pi(\beta_{s^*}(a^*))v_{s^*},$$

which shows that $C^*(\pi, v)$ is really a C^*-algebra. It is clear that $\pi \times v$ is linear. It suffices to verify the multiplicativity of $\pi \times v$ for elements of the form $a_s\delta_s$. We have

$$\begin{aligned}
(\pi \times v)(a_s\delta_s * a_t\delta_t) &= (\pi \times v)(\beta_s(\beta_{s^*}(a_s)a_t)\delta_{st}) \\
&= \pi(\beta_s(\beta_{s^*}(a_s)a_t))v_{st} \\
&= \pi(a_s)v_s\pi(a_t)v_t \\
&= (\pi \times v)(a_s\delta_s)(\pi \times v)(a_t\delta_t)
\end{aligned}$$

as desired. The following calculation shows that $\pi \times v$ preserves the *-operation:

$$\begin{aligned}(\pi \times v)(a_s\delta_s)^* &= (\pi(a_s)v_s)^* \\ &= \pi(\beta_{s^*}(a_s^*))v_{s^*} \\ &= (\pi \times v)(\beta_{s^*}(a_s^*)\delta_{s^*}) \\ &= (\pi \times v)((a_s\delta_s)^*).\end{aligned}$$

It is clear that $(\pi \times v)$ is onto. □

Definition 5.4. Let A be a C^*-algebra and β be an action of the unital inverse semigroup S on A. Define a seminorm $\|.\|_c$ on L by

$$\|x\|_c = \sup\{\|(\pi \times v)(x)\| : (\pi, v) \in \mathrm{CovRep}(A, S, \beta)\}.$$

Let $I = \{x \in L : \|x\|_c = 0\}$. The *crossed product* $A \times_\beta S$ of the C^*-algebra A and the semigroup S by the action β is the C^*-algebra gotten by the completion of the quotient L/I with respect to $\|.\|_c$. We denote the quotient map by Φ.

Since $\Phi(L)$ is dense in $A \times_\beta S$, it is clear that $\pi \times v$ induces a nondegenerate representation of $A \times_\beta S$. We denote this representation also by $\pi \times v$.

The following lemma shows that the ideal I may be nontrivial:

Lemma 5.5. *If $s, t \in S$ so that $s \leq t$, that is, $s = ft$ for some idempotent $f \in S$ then $\Phi(a\delta_s) = \Phi(a\delta_t)$ for all $a \in E_s$. In particular $\Phi(a\delta_s) = \Phi(a\delta_e)$ if s is an idempotent.*

Proof. It is clear that $a \in E_t$. If $(\pi, v) \in \mathrm{CovRep}(A, S, \beta)$ then

$$(\pi \times v)(a\delta_{ft} - a\delta_t) = \pi(a)v_f v_t - \pi(a)v_t = 0,$$

which shows $\Phi(a\delta_s - a\delta_t) = 0$. The second statement follows from the fact that $s = se$. □

If (Π, H) is a representation of $A \times_\beta S$ and $x \in L$ then we are going to write $\Pi(x)$ instead of the more precise $\Pi(\Phi(x))$.

Proposition 5.6. *Let (Π, H) be a nondegenerate representation of $A \times_\beta S$. Define a representation π of A on H by*

$$\pi(a) = \Pi(a\delta_e).$$

Let $v : S \to B(H)$ defined by

$$v_s = \operatorname*{s\text{-}lim}_\lambda \Pi(u_\lambda \delta_s)$$

where $\{u_\lambda\}$ *is an approximate identity for* E_s *and* s-lim_λ *denotes the strong operator limit. Then* $(\pi, v, H) \in \mathrm{CovRep}(A, S, \beta)$.

Proof. First we show that v_s is well defined. If $h \in \pi(E_{s^*})H$ then $h = \Pi(a\delta_e)k$ for some $a \in E_{s^*}$ and $k \in H$. So

$$\begin{aligned}\lim_\lambda \Pi(u_\lambda\delta_s)(h) &= \lim_\lambda \Pi(u_\lambda\delta_s)\Pi(a\delta_e)(k)\\ &= \lim_\lambda \Pi(u_\lambda\delta_s * a\delta_e)(k)\\ &= \lim_\lambda \Pi(\beta_s(\beta_{s^*}(u_\lambda)a)\delta_s)(k)\\ &= \Pi(\beta_s(a)\delta_s)(k),\end{aligned}$$

where we used the fact that $\beta_{s^*}(u_\lambda)$ is an approximate identity for E_{s^*}. Note that the limit is independent of the choice of $\{u_\lambda\}$ since the expression $h = \pi(a\delta_e)k$ was. On the other hand if $\langle h, \pi(E_{s^*})H\rangle = \langle h, \Pi(E_{s^*}\delta_e)H\rangle = 0$ then

$$\begin{aligned}\lim_\lambda \Pi(u_\lambda\delta_s)(h) &= \lim_\lambda \Pi(\beta_s(\beta_{s^*}(\sqrt{u_\lambda})\beta_{s^*}(\sqrt{u_\lambda}))\delta_s)(h)\\ &= \lim_\lambda \Pi(\sqrt{u_\lambda}\delta_s * \beta_{s^*}(\sqrt{u_\lambda})\delta_e)(h)\\ &= \lim_\lambda \Pi(\sqrt{u_\lambda}\delta_s)\Pi(\beta_{s^*}(\sqrt{u_\lambda})\delta_e)(h).\end{aligned}$$

But $\langle \Pi(\beta_{s^*}(\sqrt{u_\lambda})\delta_e)(h), H\rangle = \langle h, \Pi(\beta_{s^*}(\sqrt{u_\lambda})\delta_e)H\rangle = 0$ and so $\lim_\lambda \Pi(u_\lambda\delta_s)(h) = 0$. Hence v_s is well defined. It is easy to see that v_s is a bounded linear transformation. The following calculation shows that $v_s^* = v_{s^*}$:

$$\begin{aligned}v_s^* &= \text{s-}\lim_\lambda \Pi(u_\lambda\delta_s)^*\\ &= \text{s-}\lim_\lambda \Pi(\beta_{s^*}(u_\lambda)\delta_{s^*}) = v_{s^*}.\end{aligned}$$

In order to see that v_s is a partial isometry we need to show that $v_s v_s^* v_s = v_s$. It suffices to show that $v_s^* v_s h = h$ for $h \in \pi(E_{s^*})H$, since we have seen above that $(\pi(E_{s^*})H)^\perp \subset \ker v_s$. Let $h = \Pi(a\delta_e)k$ where $a \in E_{s^*}$ and $k \in H$. Using the fact that $\Phi(a\delta_{s^*s}) = \Phi(a\delta_e)$ we have

$$\begin{aligned}v_s^* v_s(h) &= \lim_\lambda \Pi(u_\lambda\delta_{s^*})\Pi(\beta_s(a)\delta_s)(k)\\ &= \lim_\lambda \Pi(\beta_{s^*}(\beta_s(u_\lambda)\beta_s(a))\delta_{s^*s})(k)\\ &= \Pi(a\delta_{s^*s})(k) = \Pi(a\delta_e)(k) = h.\end{aligned}$$

This calculation also shows that v_s has initial space $v_s^* v_s(H) = \pi(E_{s^*})H$. A similar calculation shows that v_s has final space $\pi(E_s)H$. To see that v is a semigroup

homomorphism let $h = \Pi(a\delta_e)k \in \pi(E_{t^*s^*})H$ where $a \in E_{t^*s^*}$ and $k \in H$. Then

$$\begin{aligned} v_s v_t(h) &= v_s v_t \Pi(a\delta_e)(k) \\ &= v_s \Pi(\beta_t(a)\delta_t)(k), \qquad \text{by Lemma 4.2} \\ &= \lim_\lambda \Pi(u_\lambda \delta_s)\Pi(\beta_t(a)\delta_t)(k) \\ &= \lim_\lambda \Pi(\beta_s(\beta_{s^*}(u_\lambda)\beta_t(a)\delta_{st})(k) \\ &= \Pi(\beta_s\beta_t(a)\delta_{st})(k), \qquad \text{since } \beta_t(a) \in E_{s^*} \\ &= v_{st}\Pi(a\delta_e)(k) = v_{st}(h). \end{aligned}$$

On the other hand if $\langle h, \Pi(E_{t^*s^*}\delta_e)H\rangle = 0$ then $v_{st}(h) = 0$. We show $v_s v_t(h) = 0$ as well. If w_μ is an approximate identity for E_t then

$$\begin{aligned} v_s v_t(h) &= \operatorname*{s-lim}_\lambda \Pi(u_\lambda\delta_s) \operatorname*{s-lim}_\mu \Pi(w_\mu\delta_t)(h) \\ &= \operatorname*{s-lim}_{\mu,\lambda} \Pi(u_\lambda\delta_s * w_\mu\delta_t)(h) \\ &= \operatorname*{s-lim}_{\mu,\lambda} \Pi(\beta_s(\beta_{s^*}(u_\lambda)w_\mu)\delta_{st})(h). \end{aligned}$$

By Lemma 4.2 $\beta_s(\beta_{s^*}(u_\lambda)w_\mu) \in E_{st}$, and so can be factored as xy with $x, y \in E_{st}$ (by the Cohen-Hewitt factorization theorem). Now we have

$$\Pi(xy\delta_{st})(h) = \Pi(x\delta_{st})\Pi(\beta_{t^*s^*}(y)\delta_e)(h).$$

But $\langle \Pi(\beta_{t^*s^*}(y)\delta_e)(h), H\rangle = \langle h, \Pi(\beta_{t^*s^*}(y)\delta_e)H\rangle = 0$ and so $v_s v_t(h) = 0$.

For the covariance condition, if $a \in E_{s^*}$ then

$$\begin{aligned} v_s\pi(a)v_{s^*} &= \operatorname*{s-lim}_{\lambda,\mu} \Pi(u_\mu\delta_s)\Pi(a\delta_e)\Pi(\beta_{s^*}(u_\lambda)\delta_{s^*}) \\ &= \operatorname*{s-lim}_{\lambda,\mu} \Pi(u_\mu\beta_s(a)u_\lambda\delta_{ss^*}) \\ &= \Pi(\beta_s(a)\delta_e) \end{aligned}$$

as desired. The nondegeneracy of π follows from that of Π, since $\{u_\lambda\delta_e\}$ is an approximate identity for $A \times_\beta S$ whenever $\{u_\lambda\}$ is an approximate identity for A. □

Proposition 5.7. *The correspondence $(\pi, v, H) \leftrightarrow (\pi \times v, H)$ is a bijection between* $\mathrm{CovRep}(A, S, \beta)$ *and the class of nondegenerate representations of $A \times_\beta S$.*

Proof. Let $(\tilde\pi, \tilde v, H) \in \mathrm{CovRep}(A, S, \beta)$. Let (π, v) the the covariant representation induced by $\tilde\pi \times \tilde v$. Then

$$\pi(a) = (\tilde\pi \times \tilde v)(a\delta_e) = \tilde\pi(a)v_e = \tilde\pi(a)$$

$$v_s = \text{s-}\lim_{\lambda} (\tilde{\pi} \times \tilde{v})(u_\lambda \delta_s) = \text{s-}\lim_{\lambda} \tilde{\pi}(u_\lambda)\tilde{v}_s = \tilde{v}_s.$$

The last equality holds because $\tilde{\pi}(u_\lambda)$ converges strongly to the identity on $\tilde{\pi}(E_s)H$. On the other hand if Π is a representation of $A \times_\beta S$, (π, v) is induced by Π and $a \in E_s$ then

$$\begin{aligned}(\pi \times v)(a\delta_s) &= \pi(a)v_s \\ &= \Pi(a\delta_e)\, \text{s-}\lim_{\lambda} \Pi(u_\lambda \delta_s) \\ &= \text{s-}\lim_{\lambda} \Pi(a\delta_e * u_\lambda \delta_s) \\ &= \text{s-}\lim_{\lambda} \Pi(au_\lambda \delta_s) \\ &= \Pi(a\delta_s).\end{aligned}$$

Thus the correspondence is a bijection. □

Proposition 5.8. *If β is an action of the semilattice S on a C^*-algebra A. Then $A \times_\beta S$ is isomorphic to A.*

Proof. If (π, v) is any covariant representation of β then by Lemma 5.5 $(\pi \times v)(A \times_\beta S) = \pi(A)$. In particular if π is a faithful representation of A on the Hilbert space H and v_f is the projection onto $\pi(E_f)H$ for all $f \in S$ then (π, v) is a covariant representation of β and $(\pi \times v)(A \times_\beta S) = \pi(A) \cong A$. Every representation of $A \times_\beta S$ factors through $(\pi \times v)$ for if $(\rho \times z)$ is a representation of $A \times_\beta S$ then $(\rho \times z) = \rho \circ \pi^{-1} \circ (\pi \times v)$. Thus $(\pi \times v)$ is faithful and so $A \times_\beta S$ and A must be isomorphic. □

The following example shows that unlike in the partial action case the crossed product $A \times_\beta S$ is not the enveloping C^*-algebra of L in general.

Example 5.9. Let $S = \{e, f\}$ be the unital inverse semigroup that contains the identity e and an idempotent f. Let $A = \mathbf{C}$ and β_s be the identity map of A for all $s \in S$ as in Example 4.7. It is clear that $L = l^1(S)$. Wordingham [5] shows that the left regular representation of $l^1(S)$ on $l^2(S)$ is faithful and so the enveloping C^*-algebra cannot be the same as $A \times_\beta S$, which is isomorphic to $\mathbf{C}$ by Proposition 5.8.

The next two results describe two quite different crossed products associated with an inverse semigroup itself.

Proposition 5.10. *Let S be a unital inverse semigroup, and let β_s be the identity map of $\mathbf{C}$ for all $s \in S$. Then β_s is an action of S on $\mathbf{C}$, and $\mathbf{C} \times_\beta S$ is isomorphic to the C^*-algebra of the maximal group homomorphic image of S.*

Proof. For $s, t \in S$ let $s \sim t$ if and only if there is an idempotent $f \in S$ so that $fs = ft$. Then $\sim$ is a congruence on S and $G = S/\sim$ is the maximal group homomorphic image of S. Let $[s]$ denote the equivalence class of $s \in S$.

Let $\Phi : L \to L/I$ be as in Definition 5.4. Define a *-homomorphism

$$\Psi : G \to L/I \quad \text{by} \quad \Psi([s]) = \Phi(\delta_s).$$

Ψ is well defined since if $s \sim t$ then there is an idempotent $f \in S$ so that $fs = ft$ and so by Lemma 5.5 we have $\Phi(\delta_s) = \Phi(\delta_f)\Phi(\delta_s) = \Phi(\delta_f)\Phi(\delta_t) = \Phi(\delta_t)$. Ψ extends to a *-homomorphism $\Psi : C^*(G) \to \mathbf{C} \times_\beta S$. The extension is onto since $\Psi(G) = \Phi(\delta_S)$ has dense span in $\mathbf{C} \times_\beta S$.

Going the other way, define a covariant representation of $(\mathbf{C}, S, \beta)$ in $C^*(G)$, considered to be represented on a Hilbert space, by

$$\begin{aligned} \pi &: \mathbf{C} \to C^*(G), \quad \pi(a) = ae \\ v &: S \to C^*(G), \quad v_s = [s], \end{aligned}$$

where G is identified with its canonical image in $C^*(G)$. In fact, (π, v) satisfies the covariant condition $v_s \pi(a) v_{s^*} = a[ss^*] = \pi(a)$ because all the idempotents are congruent.

The covariant representation (π, v) defines a *-homomorphism

$$(\pi \times v) : \mathbf{C} \times_\beta S \to C^*(G).$$

For $[s] \in G$ we have

$$(\pi \times v) \circ \Psi([s]) = (\pi \times v)(\delta_s) = v_s = [s].$$

So $(\pi \times v) \circ \Psi$ is the identity map on G. Since a representation of G has a unique extension to $C^*(G)$, $(\pi \times v) \circ \Psi$ must be the identity map on $C^*(G)$ and hence Ψ is an isomorphism between $C^*(G)$ and $\mathbf{C} \times_\beta S$. $\square$

Proposition 5.11. *Let S be a unital inverse semigroup with idempotent semilattice E. Define a semigroup action β of S on $C^*(E)$ so that $\beta_s : E_{s^*} \to E_s$ is determined by $\beta_s(\delta_f) = \delta_{sfs^*}$, where E_s is the closed span of the set $\{\delta_f : f \leq ss^*\}$ in $C^*(E)$. Then $C^*(S)$ is isomorphic to $C^*(E) \times_\beta S$.*

Proof. Let $\Phi : L \to L/I$ be as in Definition 5.4. We are going to identify S with its canonical image in $C^*(S)$. Define a *-homomorphism

$$\Psi : S \to L/I \quad \text{by} \quad \Psi(s) = \Phi(ss^*\delta_s).$$

In fact Ψ is a *-homomorphism since

$$\begin{aligned} \Psi(s)\Psi(t) &= \Phi(ss^*\delta_s)\Phi(tt^*\delta_t) = \Phi(\beta_s(\beta_{s^*}(ss^*)tt^*)\delta_{st}) \\ &= \Phi(\beta_s(s^*ss^*stt^*)\delta_{st}) = \Phi(ss^*stt^*s^*\delta_{st}) \\ &= \Phi(st(st)^*\delta_{st}) = \Psi(st), \end{aligned}$$

and

$$\begin{aligned}\Psi(s)^* &= \Phi(ss^*\delta_s)^* = \Phi(\beta_{s^*}((ss^*)^*)\delta_{s^*}) \\ &= \Phi(s^*ss^*s\delta_{s^*}) = \Psi(s^*).\end{aligned}$$

Ψ extends to a *-homomorphism $\Psi : C^*(S) \to C^*(E) \times_\beta S$. The extension is onto since if $f \in E$ with $f \leq ss^*$ then by Lemma 5.5 we have

$$\begin{aligned}\Phi(f\delta_s) &= \Phi(f\delta_e)\Phi(f\delta_s) = \Phi(f\delta_f)\Phi(f\delta_s) \\ &= \Phi(f\delta_{fs}) = \Phi(fs(fs)^*\delta_{fs}) \\ &= \Psi(fs),\end{aligned}$$

which means the span of $\Psi(S)$ is dense in $C^*(E) \times_\beta S$.

We show that Ψ is in fact an isomorphism. Let $\pi : C^*(E) \to C^*(S)$ be the canonical inclusion map and define $v : S \to C^*(S)$ by $v_s = s$. We show that (π, v) is a covariant representation of $(C^*(E), S, \beta)$ if $C^*(S)$ is considered to be represented on a Hilbert space. It is clear that v is a semigroup homomorphism into an inverse semigroup of partial isometries. The requirements for the initial and final spaces are satisfied since $v_s v_s^* = ss^* = \pi(p_s)$. It suffices to verify the covariance condition for elements $f \in E_{s^*}$ where $f \leq s^*s$. For such f we have

$$v_s\pi(f)v_{s^*} = sfs^* = \beta_s(f),$$

as desired.

The covariant representation (π, v) gives a *-homomorphism

$$(\pi \times v) : C^*(E) \times_\beta S \to C^*(S).$$

Since for $s \in S$ we have

$$(\pi \times v) \circ \Psi(s) = (\pi \times v)(\Phi(ss^*\delta_s)) = \pi(ss^*)v_s = ss^*s = s,$$

the composition $(\pi \times v) \circ \Psi$ is the identity map on S. Since a representation of S has a unique extension to $C^*(S)$, $(\pi \times v) \circ \Psi$ must be the identity map on $C^*(S)$, and so Ψ is an isomorphism between $C^*(S)$ and $C^*(E) \times_\beta S$. □

6. Connection between the crossed products

Let (A, G, α) be a partial action of the group G on A. Let (Π, H) be a faithful nondegenerate representation of the crossed product $A \times_\alpha G$. McClanahan [4] showed that $\Pi = \pi \times u$ for some covariant representation (π, u, H) of (A, G, α). Define, as in Proposition 4.4, a unital inverse semigroup

$$S = \{(\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n}) : g_1, \ldots, g_n \in G\}$$

and an action β of S so that $\beta_s = \alpha_{g_1} \cdots \alpha_{g_n}$ for $s = (\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n}) \in S$. We are going to show that the crossed products $A \times_\alpha G$ and $A \times_\beta S$ are canonically isomorphic. First we need the following

Lemma 6.1. *Let* $(\rho, z, K) \in \operatorname{CovRep}(A, S, \beta)$ *and define a covariant representation* (ρ, w, K) *of* (A, G, α) *by* $w_g = z(\alpha_g, u_g)$ *as in Proposition* 4.6. *Then* $(\rho\times z)(A\times_\beta S) = (\rho \times w)(A \times_\alpha G)$.

Proof. By Proposition 5.3, it suffices to show that

$$\sum_{s\in S} \rho(E_s)z_s = \sum_{g\in G} \rho(D_g)w_g.$$

Let $g \in G$ and $s = (\alpha_g, u_g)$. If $a \in D_g = E_s$ then $\rho(a)w_g = \rho(a)z_s$ and so $\sum_{s\in S} \rho(E_s)z_s \supset \sum_{g\in G} \rho(D_g)w_g$. On the other hand if $s = (\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n})$ and $a \in E_s = D_{g_1} D_{g_1 g_2} \cdots D_{g_1\cdots g_n}$ then by Corollary 3.9 we have

$$\begin{aligned} \rho(a)z_s &= \rho(a)z(\alpha_{g_1}, u_{g_1}) \cdots z(\alpha_{g_1}, u_{g_n}) \\ &= \rho(a)w_{g_1} \cdots w_{g_n} = \rho(a)w_{g_1\cdots g_n} \end{aligned}$$

and so $\sum_{s\in S} \rho(E_s)z_s \subset \sum_{g\in G} \rho(D_g)w_g$. □

Now we can prove our main result.

Theorem 6.2. *Let* α *be a partial action of a group* G *on the* C^**-algebra* A*, and let* (π, u, H) *be a covariant representation of* (A, G, α) *so that* $\pi \times u$ *is a faithful representation of* $A \times_\alpha G$*. Define an inverse semigroup by*

$$S = \{(\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n}) : g_1, \ldots, g_n \in G\}$$

and an action β *of* S *by* $\beta_s = \alpha_{g_1} \cdots \alpha_{g_n}$ *for* $s = (\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n})$*. Then the crossed products* $A \times_\alpha G$ *and* $A \times_\beta S$ *are isomorphic.*

Proof. Let $v_s = u_{g_1} \cdots u_{g_n}$ for $s = (\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n})$. We know from Proposition 4.6 that $(\pi, v, H) \in \operatorname{CovRep}(A, S, \beta)$. Note that $C^*(\pi, u) = C^*(\pi, v)$. It suffices to show that $\pi \times v$ is a faithful representation of $A \times_\beta S$ because then by Lemma 6.1 $(\pi \times v)^{-1} \circ (\pi \times u) : A \times_\alpha G \to A \times_\beta S$ is an isomorphism. To show this consider another representation of $A \times_\beta S$, which by Proposition 5.7 must be in the form $\rho \times z$ for some $(\rho, z) \in \operatorname{CovRep}(A, S, \beta)$. By Proposition 4.6 the definition $w_g = z(\alpha_g, u_g)$ gives a covariant representation (ρ, w, K) of (A, G, α). By Lemma 6.1 $C^*(\rho, w) = C^*(\rho, z)$. Since $\pi \times u$ is a faithful representation, there is a homomorphism Θ making the following diagram commute:

$$\begin{tikzcd} & A \times_\alpha G \arrow[dl, "\pi\times u"'] \arrow[dr, "\rho\times w"] & \\ C^*(\pi, u) \arrow[rr, "\Theta"] & & C^*(\rho, w) \end{tikzcd}$$

We are going to show that the diagram

$$\begin{array}{ccc} & A \times_\beta S & \\ \swarrow_{\pi\times v} & & {}_{\rho\times z}\searrow \\ C^*(\pi, v) & \xrightarrow{\Theta} & C^*(\rho, z) \end{array}$$

commutes, which is going to prove that $\pi \times v$ is faithful. Since finite sums of elements of the form $a\delta_s \in L$ (more precisely the images under Φ) are dense in $A \times_\beta S$, it suffices to check that

$$\Theta \circ (\pi \times v)(a\delta_s) = (\rho \times z)(a\delta_s)$$

where $s = (\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n})$ and $a \in E_s = D_{g_1} D_{g_1 g_2} \cdots D_{g_1 \cdots g_n}$. But this is true by the following calculation:

$$\begin{aligned} \Theta((\pi \times v)(a\delta_s)) &= \Theta(\pi(a)v_s) \\ &= (\rho \times w) \circ (\pi \times u)^{-1}(\pi(a)u_{g_1} \cdots u_{g_n}) \\ &= (\rho \times w) \circ (\pi \times u)^{-1}(\pi(a)u_{g_1 \cdots g_n}) \\ &= (\rho \times w)(a\delta_{g_1 \cdots g_n}) \\ &= \rho(a)w_{g_1 \cdots g_n} \\ &= \rho(a)w_{g_1} \cdots w_{g_n} \\ &= \rho(a)z(\alpha_{g_1}, u_{g_1}) \cdots z(\alpha_{g_n}, u_{g_n}) \\ &= \rho(a)z_s \\ &= (\rho \times z)(a\delta_s), \end{aligned}$$

where we have appealed to Corollary 3.9 twice more. □

Example 6.3. Let $A = \mathbf{C}^2$, $G = \mathbf{Z}$, $D_0 = A$, $D_{-1} = \{(a, 0) : a \in A\}$, $D_1 = \{(0, a) : a \in A\}$ and $D_n = \{(0, 0)\}$ for $n \in G \setminus \{-1, 0, 1\}$. Let α_0 be the identity map α_1 be the forward shift $\alpha_1(a, 0) = (0, a)$ and define $\alpha_n = \alpha_1^n$ for $n \neq 0$. Then $A \times_\alpha G$ is isomorphic to the matrix algebra M_2 [2], [4, Example 2.5].

We construct a faithful representation $\pi \times u$ of the partial crossed product $A \times_\alpha G$. Let π be the representation of A on the Hilbert space $H = \mathbf{C}^2$ by multiplication operators, that is,

$$\pi(a_1, a_2)(h_1, h_2) = (a_1 h_1, a_2 h_2)$$

for $(a_1, a_2) \in A$ and $(h_1, h_2) \in H$. Let u_1 be the forward shift, u_{-1} the backward shift on H, and let u_n be the constant zero map for all $n \in G \setminus \{-1, 0, 1\}$.

The unital inverse semigroup S generated by $\{(\alpha_n, u_n) : n \in G\}$ contains six elements

$$S = \{e, f, s, s^*, s^*s, ss^*\}$$

where $e = 1_H$ is the identity of S, the zero element f of S is the constant zero map and $s = (\alpha_1, u_1)$. Let $E_e = A$, $E_f = \{(0,0)\}$, $E_{s^*} = E_{s^*s} = D_{-1}$ and $E_s = E_{ss^*} = D_1$. Define the semigroup action β of S as in Proposition 4.4. Then β_s is the forward shift, β_{s^*} is the backward shift and β_t is the identity map for all other $t \in S$. As we have seen in Theorem 6.2 the crossed product $A \times_\beta S$ is isomorphic to the matrix algebra M_2. □

Notice that in the last example the semigroup S is isomorphic to the inverse semigroup generated by $\{\alpha_n : n \in \mathbf{N}\}$ as well as to the inverse semigroup generated by $\{u_n : n \in \mathbf{N}\}$. Based upon experience with group actions, it might seem natural to expect that S is isomorphic to the inverse semigroup generated by the range of u whenever $\pi \times u$ is a faithful representation of $A \times_\alpha G$. Perhaps surprisingly this is not the case. All three semigroups can be non-isomorphic as the following example shows.

Example 6.4. Let $A = C[0,1]$, $G = \mathbf{Z}_2$, $D_0 = A$, and let α_1 be the identity map on $D_1 = \{x \in A : x(1) = 0\}$. We construct a faithful representation $\pi \times u$ of the partial crossed product $A \times_\alpha G$. Let π be the representation of A on the Hilbert space $L^2[0,1] \times L^2[0,1]$ defined by $\pi(f) = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$ and let $u_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $u_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By [4, Propositions 3.4 and 4.2] $\pi \times u$ is faithful since $\mathbf{Z}_2$ is amenable. The inverse semigroup generated by $\{u_0, u_1\}$ is isomorphic to $\mathbf{Z}_2$. It is clear that $\{\alpha_0, \alpha_1\}$ is a semilattice, hence is definitely not isomorphic to the inverse semigroup $\{u_0, u_1\}$. The inverse semigroup generated by $\{(\alpha_0, u_0), (\alpha_1, u_1)\}$ contains three elements $\{(\alpha_0, u_0), (\alpha_1, u_1), (\alpha_1, u_0)\}$. □

Although every partial crossed product is isomorphic to a crossed product by an action of a unital inverse semigroup, this semigroup action may not be unique up to isomorphism. For all we know different faithful representations $\Pi = \pi \times u$ of the crossed product $A \times_\alpha G$ could generate essentially different semigroup actions. If we want to talk about a canonical semigroup action associated with $A \times_\alpha G$ then we can choose Π to be the universal representation of $A \times_\alpha G$.

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